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Exact solution of the integrable *XXZ* Heisenberg model with arbitrary spin: II. Thermodynamics of the system

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Abstract. The thermodynamics of the integrable generalisation of the *XXZ* Heisenberg model with arbitrary spin is studied. The low- and high-temperature heat capacities and magnetic susceptibility are computed.

1. Introduction

In the previous paper (Kirillov and Reshetikhin 1987) we obtained the thermodynamic equations of the *XXZ* model with spin S. This model describes an anisotropic spin-S chain with nearest-neighbour interactions. The Hamiltonian of the model was given in the paper mentioned above.

Let us define the following sequences starting from the anisotropy parameter γ (see Takahashi and Suzuki 1972)

$$p_0=\pi/\gamma \qquad p_1=1 \qquad b_i=\left[\frac{p_i}{p_{i+1}}\right] \qquad p_{i+1}=p_{i-1}-b_{i-1}p_i \qquad i\geqslant 1 \tag{1.1}$$

$$y_{-1}=0 \qquad y_0=1 \qquad y_1=b_0 \qquad y_{i+1}=y_{i-1}+b_iy_i \qquad i\geqslant 0 \tag{1.2}$$

$$m_0=0 \qquad m_{i+1}=m_i+b_i \qquad i\geqslant 0 \tag{1.3}$$

$$n_j=y_{i-1}+(j-m_i)y_i \qquad r(j)=i \qquad m_i\leqslant j<m_{i+1} \tag{1.4}$$

$$q_j=(-1)^{r(j)}(p_i-(j-m_i)p_{i+1}) \qquad m_i\leqslant j<m_{i+1}. \tag{1.5}$$

The free energy of the system is expressed by

$$F(H,T)=-\sum_{j\geqslant 1}\int_{-\infty}^{+\infty}(-1)^{r(j)}a_{j,2S}(\lambda)T\log[1+\exp(-\beta\varepsilon_j(\lambda))]\,\mathrm{d}\lambda \tag{1.6}$$

where $\varepsilon_j(\lambda)$ are the solutions of the following system:

$$2p_0\varepsilon a_{j,2S}(\lambda)+n_jH-T\log[1+\exp(\beta\varepsilon_j)]+\sum_{k\geqslant 1}A_{jk}*T\log[1+\exp(-\beta\varepsilon_k)]=0. \tag{1.7}$$

Here $*$ is the convolution of functions, H is the magnetic field, T is the temperature and $\varepsilon=\pm 1$. The functions $a_{j,2S}(\lambda)$ and $A_{jk}(\lambda)$ are given by their Fourier transforms:

$$\hat{A}_{jk}(x)=\hat{A}_{kj}(x)=2\hat{a}_k(x)\hat{n}_j(x)+(-1)^{r(k)}\delta_{k,m_{\alpha+1}}\delta_{j,m_{\alpha+1}-1} \qquad k\geqslant j \tag{1.8}$$

$$\hat{a}_{j,2S}(x)=\hat{A}_{j,\sigma-1}(x)\hat{S}_{r+1}(x)+2\cosh(q_\sigma x)\sum_{p=1}^{r}\hat{A}_{jm_l-1}(x)\hat{S}_l(x)\hat{S}_{l+1}(x) \tag{1.9}$$

where

$$\hat{S}_i(x)=\frac{1}{2\cosh(p_i x)} \tag{1.10}$$

$$\hat{a}_j(x)=\frac{\sinh(q_j x)}{\sinh(p_0 x)} \tag{1.11}$$

$$\hat{n}_j(x)=\cos\left[\left(\left\{\frac{n_j}{p_0}\right\}-\frac{1-(-1)^{r(j)}}{2}\right)p_0 x\right]+\sum_{l=1}^{n_j-1}\cosh\left[\left(\left\{\frac{n_j-l}{p_0}\right\}-\left\{\frac{l}{p_0}\right\}\right)p_0 x\right] \tag{1.12}$$

where σ is connected with the value of spin by $1+2S=n_\sigma$.

We use the following normalisation of the Fourier transform

$$f(\lambda)=\int_{-\infty}^{+\infty}\exp(\mathrm{i}\lambda x)\hat{f}(x)\frac{\mathrm{d}x}{2\pi}\qquad \hat{f}(x)=\int_{-\infty}^{+\infty}\exp(-\mathrm{i}\lambda x)f(\lambda)\,\mathrm{d}\lambda. \tag{1.13}$$

Our aim in the present work is to study the thermodynamic equations described above. Following the methods developed by Takahashi and Suzuki (1972), Tsvelick and Wiegmann (1983) and Yang and Yang (1966) we compute the asymptotics of the heat capacities in the low- and high-temperature limits and magnetic susceptibility in a small magnetic field.

2. Computation of the specific heat in the low- and high-temperature limits

Let us start by transforming our basic system (1.7) in a way that is convenient for our further computations. To this end divide the set of indices j in (1.7) into three groups:

$$\varepsilon=(-1)^r\begin{cases}\{j_0\}=\{m_i\,|\,i=r+1(\mathrm{mod}\,2),\ i\leqslant r+1\}\\ \{j_1\}=\{m_i\leqslant j<m_i\,|\,i=r+1(\mathrm{mod}\,2),\ i\leqslant r+1\}\\ \{j_2\}=\{1\leqslant j<m_{\alpha+1}-1\,|\,j\notin\{j_0\}\cup\{j_1\}\}\end{cases}$$
$$\varepsilon=(-1)^{r+1}\begin{cases}\{j_0\}=\{m_i,\ \sigma-1\,|\,i=r(\mathrm{mod}\,2),\ i\leqslant r\}\\ \{j_1\}=\{m_{i-1}\leqslant j<m_i\,|\,i=r(\mathrm{mod}\,2),\ i\leqslant r\}\\ \{j_2\}=\{1\leqslant j<m_{\alpha+1}-1\,|\,j\notin\{j_0\}\cup\{j_1\}\}.\end{cases} \tag{2.1}$$

Recall that indices label the so-called strings which describe the solutions of Bethe's equation in the thermodynamic limit. The division (2.1) is in accordance with the role of different strings in the composition of the ground state and excitations over it.

We regard (1.7) as a system of equations for the functions $\varepsilon_{j_0}(\lambda)$, $\varepsilon_{j_1}(\lambda)$, $\varepsilon_{j_2}(\lambda)$. In order to study their low-temperature behaviour we first invent the kernel $A_{k_0j_0}$ and then invert the kernel $A^{(2,2)}_{k_2j_2}$ in the resulting system. We obtain the following equations:

$$\frac{1}{T}\varepsilon^{(0)}_{j_0}-\delta_{j_0,\varkappa}\frac{zH}{2T}$$
$$=\sum_{k_0}B_{j_0k_0}*\log(1+\exp(\beta\varepsilon_{k_0}))+\log[1+\exp(-\beta\varepsilon_{j_0})]$$
$$-\sum_{k_1}A^{(0,1)}_{j_0k_1}*\log[1+\exp(-\beta\varepsilon_{k_1})]-\sum_{k_2}A^{(0,2)}_{j_0k_2}*\log[1+\exp(-\beta\varepsilon_{k_2})] \tag{2.2}$$

$$\frac{1}{T}\varepsilon_{j_1}^{(0)}=\log[1+\exp(\beta\varepsilon_{j_1})]+\sum_{k_1}A_{j_1k_1}^{(1,1)}*\log[1+\exp(-\beta\varepsilon_{k_1})]$$
$$-\sum_{k_0}A_{j_1k_0}^{(1,0)}*\log[1+\exp(\beta\varepsilon_{k_0})]-\sum_{k_2}A_{j_1k_2}^{(1,2)}*\log[1+\exp(-\beta\varepsilon_{k_2})] \qquad (2.3)$$

$$0=\sum_{k_2}B_{j_2k_2}^{(2,2)}*\log[1+\exp(\beta\varepsilon_{k_2})]+\log[1+\exp(-\beta\varepsilon_{j_2})]$$
$$+\sum_{k_1}B_{j_2k_1}^{(2,1)}*\log[1+\exp(-\beta\varepsilon_{k_1})]-\sum_{k_0}B_{j_2k_0}^{(2,0)}*\log[1+\exp(\beta\varepsilon_{k_0})]. \qquad (2.4)$$

Here $\beta=1/T$, $\varkappa=m_{r+1}$ if $\varepsilon=(-1)^r$ and $\varkappa=\sigma-1$ if $\varepsilon=(-1)^{r+1}$ and the kernels $B^{(\alpha,\beta)}$ are given in appendix 1. Explicit formulae for $A^{(\alpha,\beta)}$ and $\varepsilon_j^{(0)}$ (which are the energies of the j strings) are given in our previous paper (Kirillov and Reshetikhin 1987). The value of z determines the spin renormalisation

$$z=(-1)^{r(\varkappa)}p_0/q_\varkappa. \qquad (2.5)$$

Using the system (2.2)-(2.4) the expression for the free energy (1.6) may be transformed to the following form:

$$F(H,T)=\mathscr{E}_0(0)-\sum_{j_0}\int_{-\infty}^{+\infty}\varepsilon_{j_0}^{(0)}(\lambda)T\log[1+\exp(\beta\varepsilon_{j_0}(\lambda)]\,\mathrm{d}\lambda$$
$$-\sum_{j_2}\int_{-\infty}^{+\infty}\varepsilon_{j_2}^{(0)}(\lambda)T\log[1+\exp(-\beta\varepsilon_{j_2}(\lambda))]\,\mathrm{d}\lambda \qquad (2.6)$$

where $\mathscr{E}_0(0)$ is the ground-state energy at $H=0$ (see Kirillov and Reshetikhin 1987).

Let us consider first the low-temperature limit in which $H\to 0$, $T\to 0$ and $y=H/T$ is fixed. We introduce the functions (see Babujian 1982, Tsvelick and Wiegmann 1983)

$$\varphi_j(\lambda)=\frac{1}{T}\varepsilon_j\left(\lambda-\frac{1}{x_i}\log T\right) \qquad m_{i-2}\leqslant j<m_i \qquad (2.7)$$

where at $\varepsilon=(-1)^r$ we have $x_i=\pi/2p_i$ for (mod 2), $i\leqslant r+1$ and $x_i=\pi/2p_{r+2}$ for $i\geqslant r+3$, at $\varepsilon=(-1)^{r+1}$ we have $x_i=\pi/2p_i$ for $i=r(\text{mod }2)$, $i\leqslant r$, and $x_i=\pi/2p_{r+1}$ for $i\geqslant r+2$.

From (2.2)-(2.4) we obtain a set of non-linear integral equations for the functions $\varphi_j(\lambda)$ which depend only on y at $T\to 0$. This system is almost of the same form as the system (2.2)-(2.4), the difference being that H/T and $\varepsilon_j^{(0)}(\lambda-(1/x_i)\log T)/T$ are replaced by y and $l_j(\lambda)$, respectively. Here we denote by $l_j(\lambda)$ the asymptotics of $(1/T)\varepsilon_j^{(0)}[\lambda-(1/x_i)\log T]$ at $T\to 0$:

$$l_j(\lambda)=4p_0\cos\left(\frac{\pi q_\sigma}{2p_i}\right)\sin\left((j-m_i+1)\frac{\pi p_{i+1}}{2p_i}\right)\exp\left(-\frac{\pi\lambda}{2p_i}\right)$$
$$\times\left[p_i\sinh\left(\frac{\pi p_{i+1}}{2p_i}\right)\right]^{-1} \qquad m_i\leqslant j<m_{i+1} \qquad (2.8)$$

$$l_{m_{i+1}}(\lambda)=2p_0\cos\left(\frac{\pi q_\sigma}{2p_i}\right)\exp\left(\frac{-\pi\lambda}{2p_i}\right)\left[p_i\sin\left(\frac{\pi p_{i+1}}{2p_i}\right)\right]^{-1} \qquad (2.9)$$

$$l_{\sigma-1}(\lambda)=2\frac{p_0}{p_{r+1}}\exp\left(-\frac{\pi\lambda}{2p_{r+1}}\right). \qquad (2.10)$$

Here $i=r+1(\text{mod }2)$, $i\leqslant r+1$ if $\varepsilon=(-1)^r$ and $i=r(\text{mod }2)$, $i\leqslant r$ if $\varepsilon=(-1)^{r+1}$.

Substituting (2.7) into the formula (2.6) we obtain the low-temperature asymptotics of the free energy:

$$F(H, T)-\mathscr{E}_0(0)$$

$$=-T^2\sum_{j_0}\int_{-\infty}^{+\infty} l_{j_0}(\lambda)T\log[1+\exp(\varphi_{j_0}(\lambda))]\,\mathrm{d}\lambda$$

$$-T^2\sum_{j_1}\int_{-\infty}^{+\infty} l_{j_2}(\lambda)\log[1+\exp(-\varphi_{j_1}(\lambda))]\,\mathrm{d}\lambda. \tag{2.11}$$

We can now obtain the specific heat of the system by using the general formulae

$$C_H=-T\left(\frac{\partial S}{\partial T}\right)_H=-T\left(\frac{\partial^2 F}{\partial T^2}\right)_H. \tag{2.12}$$

Unfortunately, these formulae are not effective at finite y. We cannot solve explicitly a set of non-linear integral equations in this case. However, the leading term of the low-temperature asymptotics of C_H may be calculated exactly at $H=0$. To this end, let us find the low-temperature asymptotics of the entropy which are given by the following expression:

$$S=\sum_{j\geqslant 1}\int_{-\infty}^{+\infty}[(\rho_j+\rho_j^{\mathrm{h}})\log(\rho_j+\rho_j^{\mathrm{h}})-\rho_j\log\rho_j-\rho_j^{\mathrm{h}}\log\rho_j^{\mathrm{h}}]\,\mathrm{d}\lambda. \tag{2.13}$$

A comparison of equations (4.23), (4.31) and (4.32) from the paper of Kirillov and Reshetikhin (1987) and (2.2)-(2.4) yields at $|\lambda|\to\infty$ the following asymptotic relations:

$$\rho_j(\lambda)\simeq(-1)^{r(j)}\frac{1}{2p_0x_i}\frac{\mathrm{d}\varepsilon_j(\lambda)}{\mathrm{d}\lambda}f\left(\frac{\varepsilon_j(\lambda)}{T}\right) \tag{2.14}$$

$$\rho_j^{\mathrm{h}}(\lambda)\simeq(-1)^{r(j)}\frac{1}{2p_0x_i}\frac{\mathrm{d}\varepsilon_j(\lambda)}{\mathrm{d}\lambda}\left[1-f\left(\frac{\varepsilon_j(\lambda)}{T}\right)\right] \tag{2.15}$$

where $f(\varepsilon)=(1+\mathrm{e}^{\varepsilon})^{-1}$ is the Fermi function and x_i have been defined earlier.

Substituting these relations into the expression for the entropy we obtain the leading term of its asymptotics as $T\to 0$

$$S=-\sum_{j\geqslant 1}(-1)^{r(j)}\frac{T}{2p_0x_i}\int_{\varphi_j(-\infty)}^{\varphi_j(\infty)}[f(\varphi)\log f(\varphi)+(1-f(\varphi))\log(1-f(\varphi))]\,\mathrm{d}\varphi \tag{2.16}$$

where the functions $\varphi_j(\lambda)$ are defined by (2.7).

Let us introduce the notation $b_j=\varphi_j(\infty)$, $c_j=\varphi_j(-\infty)$ and replace the integration over $\mathrm{d}\varphi$ in (2.16) by integration over $\mathrm{d}f(\varphi)$. Now the expression (2.16) can be rewritten in terms of the dilogarithmic Rogers function (see, for example, Levin 1958)

$$L(x)=-\frac{1}{2}\int_0^x\left(\frac{\log y}{1-y}+\frac{\log(1-y)}{y}\right)\mathrm{d}y \tag{2.17}$$

$$S=T\sum_{j\geqslant 1}\frac{1}{p_0x_i}(-1)^{r(j)}[L(f(b_j))-L(f(c_j))]. \tag{2.18}$$

The constants b_j and c_j are determined from the system (2.2)-(2.4). To describe them it is useful to introduce a standard sequence $b_j(p_0)$ which is defined by

$$f(b_j(p_0))=\left(\frac{y_i}{n_j+y_i}\right)^2 \qquad 1\leqslant j\leqslant m_{\alpha+1}-2 \tag{2.19}$$

$$f(b_{m_{\alpha+1}-1}(p_0))=\frac{y_\alpha}{y_{\alpha+1}} \qquad f(b_{m_{\alpha+1}}(p_0))=1-\frac{y_\alpha}{y_{\alpha+1}} \tag{2.20}$$

where p_0 is a fixed rational number and the numbers n_j and y_i are given by (1.1)-(1.5).

The set b_j are the standard sequence for $p_0=\pi/\gamma$, $b_i=b_i(\pi/\gamma)$. The set c_j depends on the structure of the Dirac sea. If $\varepsilon=(-1)^r$ then we get the following expression for $f(c_j)$:

$$f(c_{m_i})=1 \qquad i=r+1(\mathrm{mod}\,2) \qquad i\leqslant r+1 \tag{2.21}$$

$$f(c_j)=0 \qquad m_{i-1}\leqslant j<m_i \tag{2.22}$$

$$f(c_j)=\sin^2\left(\frac{\pi}{b_{i-2}+2}\right)\left[\sin^2\left(\frac{\pi(j+1)}{b_{i-2}+2}\right)\right]^{-1} \qquad m_{i-2}<j<m_{i-1} \tag{2.23}$$

$$f(c_j)=f(b_{j-m_{r+1}}(\tilde{p}_0)) \qquad j>m_{r+1} \qquad \tilde{p}_0=\frac{p_{r+1}}{p_{r+2}}. \tag{2.24}$$

If $\varepsilon=(-1)^{r+1}$ the sequence $f(c_j)$ is slightly different:

$$f(c_{m_i})=1 \qquad i=r(\mathrm{mod}\,2) \qquad i\leqslant r \tag{2.25}$$

$$f(c_{\sigma-1})=1 \qquad f(c_j)=0 \qquad m_{i-1}\leqslant j<m_i \tag{2.26}$$

$$f(c_j)=\sin^2\left(\frac{\pi}{b_{i-2}+2}\right)\left[\sin^2\left(\frac{\pi(j+1)}{b_{i-2}+2}\right)\right]^{-1} \qquad m_{i-2}<j<m_{i-1} \tag{2.27}$$

$$f(c_j)=\sin^2\left(\frac{\pi}{\sigma+1-m_r}\right)\left[\sin^2\left(\frac{\pi(j+1)}{\sigma+1-m_r}\right)\right]^{-1} \qquad m_r<j<\sigma-1 \tag{2.28}$$

$$f(c_j)=f(b_{j-\sigma+1}(\tilde{p}_0)) \qquad j>\sigma-1 \qquad \tilde{p}_0=(-1)^r\frac{q_{\sigma-1}}{p_{r+1}}. \tag{2.29}$$

The function $L(x)$ is known to satisfy the functional equations

$$L(x)+L(1-x)=\tfrac{1}{6}\pi^2 \qquad 0<x<1 \tag{2.30}$$

$$L(x)+L(y)=L(xy)+L\left(\frac{x(1-y)}{1-xy}\right)+L\left(\frac{y(1-x)}{1-xy}\right). \tag{2.31}$$

They imply that the following sum is equal to zero:

$$\sum_{j\geqslant 1}(-1)^{r(j)}L(f(b_j(p_0)))=0. \tag{2.32}$$

In appendix 2 another useful sum is calculated:

$$\sum_{k=2}^{n-2}L\left(\frac{\sin^2(\pi/n)}{\sin^2(\pi k/n)}\right)=\frac{n-3}{n}\frac{\pi^2}{6}. \tag{2.33}$$

Using these formulae we obtain from (2.18) an explicit expression for the low-temperature asymptotics of the entropy:

$$S = \pi T \begin{cases} \displaystyle\sum_{\substack{i \leq r+1 \\ i = r+1 (\mathrm{mod} 2)}} \frac{p_i}{p_0} \frac{b_{i-2}}{b_{i-2}+2} & \varepsilon = (-1)^r \qquad (2.34) \\ \displaystyle\sum_{\substack{i \leq r \\ i \leq r (\mathrm{mod} 2)}} \frac{p_i}{p_0} \frac{b_{i-2}}{b_{i-2}+2} + \frac{p_{r+1}}{p_0} \frac{\sigma - 1 - m_r}{\sigma + 1 - m_r} & \varepsilon = (-1)^{r+1}. \qquad (2.35) \end{cases}$$

These formulae hold whenever $r = 1$, $\varepsilon = -1$. In that case we have simply

$$S = \pi T (p_0 - 1)/3p_0. \tag{2.36}$$

Substituting the low-temperature asymptotics of the entropy into (2.12) we obtain the specific heat C_H at $T \to 0$, $H = 0$.

The structure of these expressions is quite natural from the point of view of Fermi liquid theory (see, for example, Tsvelick and Wiegmann 1983). In the case when the Dirac sea is filled by strings of different sorts, there is a corresponding number of Fermi liquids with different sound velocities (see formula (4.14) from the paper by Kirillov and Reshetikhin (1987)). To each sort of Fermi liquid there corresponds a contribution of fixed length strings to the Dirac sea. The low-temperature specific heat is simply the sum of the specific heats of all Fermi liquids.

In the paper by Babujian (1982), the following formula was obtained for low temperature asymptotics of the entropy for the isotropic Heisenberg model with higher spin S:

$$S = \frac{2T}{3} + \frac{4T}{\pi^2} \sum_{k=2}^{2S-2} L\left(\frac{\sin^2[\pi/(2S+2)]}{\sin^2[\pi k/(2S+2)]}\right). \tag{2.37}$$

Using (2.34) we calculate the exact value of this sum

$$S = \frac{2S}{S+1} T. \tag{2.38}$$

The Hamiltonian of the isotropic Heisenberg model may be obtained from the Hamiltonian of the XXZ model we are considering in the limit $p_0 \to \infty$ at $\varepsilon = -1$, $r = 0$. In this limit formula (2.38) follows from (2.35) modulo the normalisation factor $\pi/2p_0$, which is due to a different normalisation of the Hamiltonian adopted by Babujian (1982).

In order to investigate the high-temperature limit it is convenient to use equations (1.6) and (1.7). Let us consider the limit $T \to \infty$, $H \to \infty$ for fixed ratio H/T. It is not difficult to find the leading term of asymptotics for the solution of the system (1.7) in this limit (see Takahashi and Suzuki 1972)

$$\eta_j(\lambda) = \exp(\beta \varepsilon_j(\lambda)) = \eta_j^{(0)} + \mathrm{O}(1/T) \tag{2.39}$$

$$1 + \eta_j^{(0)} = \frac{\sinh[(H/2T)(n_j + y_i)]}{\sinh[(H/2T)y_i]}. \tag{2.40}$$

After substitution of these results into (1.6) we obtain

$$F(H, T) = -\tfrac{1}{2} T \sum_{i=1}^{r} \log(1 + \eta_{m_i}^{(0)}) - \tfrac{1}{2} T \log(1 + \eta_{\sigma-1}^{(0)}) + \mathrm{O}(1). \tag{2.41}$$

Using (1.4) we obtain for the free energy the following simple answer:

$$F(H, T) = -T \log\left(\frac{\sinh[(H/2T)n_\sigma]}{\sinh(H/2T)}\right) + O(1) \qquad \varepsilon = \pm 1. \tag{2.42}$$

This formula, together with (2.12), gives the high-temperature asymptotics of the specific heat.

Let us point out that (2.42) at $H \ll T$ gives the following expression for the entropy:

$$\sigma = NS = N \log(1+2S). \tag{2.43}$$

This result is naturally interpreted as the completeness of the Bethe vector multiplet. The number of Bethe vectors at $T \to \infty$ is equal to $\exp(\sigma) = (2S+1)^N$ in the thermodynamical limit.

3. The magnetic susceptbility at small magnetic field

In this section we compute the magnetic susceptbility in the XXZ model at small magnetic field. We shall consider only the bases when the Dirac sea consists of only one sort of string. It is easy to see that this is the case only when ε and r have the following values:

$$\begin{aligned} &\text{(i)} \qquad \varepsilon = 1, r = 0 \\ &\text{(ii)} \qquad \varepsilon = -1, r = 0 \\ &\text{(iii)} \qquad \varepsilon = 1, r = 1. \end{aligned} \tag{3.1}$$

From (2.6) at $T \to 0$ we obtain a useful expression for the ground-state energy

$$\mathscr{E}_0(H) - \mathscr{E}_0(0) = -\sum_{j_0} \int_{|\mu| > B_{j_0}} \varepsilon_{j_0}^{(0)}(\lambda)\varepsilon_{j_0}(\lambda)\, d\lambda. \tag{3.2}$$

Here j_0 denotes the different sorts of sea strings, B_{j_0} are defined by the conditions

$$\varepsilon_{j_0}(B_{j_0}) = 0 \tag{3.3}$$

and $\varepsilon_{j_0}^{(0)}(\lambda)$ are the energies of the j_0 strings (see Kirillov and Reshetikhin 1987). The functions $\varepsilon_{j_0}(\lambda)$ are the solutions of the system (2.2) at $T = 0$. In our case there is only one term on the right-hand side and $j_0 = m_1$, $\sigma - 1$ or m_2.

The equation for the function $\varepsilon_{j_0}(\lambda)$ has the following form:

$$-\varepsilon_{j_0}^{(0)}(\lambda) + \tfrac{1}{2}zH - \int_{|\mu| > B} J_{j_0}(\lambda - \mu)\varepsilon_{j_0}(\mu)\, d\mu = \varepsilon_{j_0}(\lambda) \tag{3.4}$$

where z is the renormalisation of the spin:

$$\text{(i)}\ z = p_0 \qquad \text{(ii)}\ z = \frac{p_0}{p_0 - \sigma + 1} \qquad \text{(iii)}\ z = \frac{p_0}{p_2} \tag{3.5}$$

and the Fourier transform of the $\varepsilon_{j_0}^{(0)}(\lambda)$ and $1 + J_{j_0} = B_{j_0 j_0}^{(0,0)}$ are given in our previous paper:

$$1 + \hat{J}_{m_1}(x) = \frac{\sinh(p_0 x)}{2 \sinh x \cosh[(p_0 - 1)x]} \qquad 1 + \hat{J}_{m_2}(x) = \frac{\sinh x \sinh(p_0 x)}{2 \sinh(p_2 x) \sinh(b_0 x) \cosh x}$$

$$1+\hat{J}_{\sigma-1}(x)=\frac{\sinh(p_0x)\sinh x}{2\sinh[(p_0-\sigma+1)x]\sinh[(\sigma-1)x]\cosh x} \tag{3.6}$$

$$\hat{\varepsilon}^{(0)}_{m_1}(x)=2p_0\frac{\sinh[(\sigma-1)x)]}{\sinh x\cosh[(p_0-1)x]}$$

$$\hat{\varepsilon}^{(0)}_{\sigma-1}(x)=\frac{4p_0}{\cosh x} \qquad \hat{\varepsilon}^{(0)}_{m_2}(x)=2p_0\frac{\sinh[(\sigma-m_1)p_2x]}{\sinh(p_2x)\cosh x}. \tag{3.7}$$

We are interested in the limit $H\to 0$ in which $B\to\infty$. In this limit equation (3.4) turns to the Wiener-Hopf type equation for the function $\varepsilon_{j_0}(\lambda+B)$ (see Yang and Yang 1966):

$$-\varepsilon^{(0)}_{j_0}(\lambda+B)+\tfrac{1}{2}zH-\int_0^\infty J_{j_0}(\lambda-\mu)\varepsilon_{j_0}(\mu+B)\,\mathrm{d}\mu=\varepsilon_{j_0}(\lambda+B). \tag{3.8}$$

Let us introduce the functions

$$\hat{\varepsilon}^{\pm}_{j_0}(x)=\pm\int_0^\infty \exp(\mathrm{i}\lambda x)\varepsilon_{j_0}(\lambda+B)\,\mathrm{d}\lambda. \tag{3.9}$$

Using the Wiener-Hopf method one can obtain the function $\hat{\varepsilon}^{+}_{j_0}(x)$ explicitly:

$$\hat{\varepsilon}^{+}_{j_0}(x)=G_+(x)\left(-\int_{-\infty}^{+\infty}\frac{\mathrm{d}x'}{2\pi}\,\frac{G_-(x')\hat{\varepsilon}^{(0)}_{j_0}(x')\exp(-\mathrm{i}Bx')}{x'-x-\mathrm{i}0}+\frac{z}{2}\,\frac{\mathrm{i}HG_-(0)}{x+\mathrm{i}0}\right). \tag{3.10}$$

Here the functions $G_\pm(x)$ are analytic and nowhere zero in the upper (lower) half-plane normalised by the condition $G_\pm(\infty)=1$ and are the solutions of the factorisation problem

$$1+\hat{J}_{j_0}(x)=G_+^{-1}(x)G_-^{-1}(x) \qquad G_-(x)=G_+(-x). \tag{3.11}$$

The main contribution to the integral in (3.10) at $B\to\infty$ is given by the pole $\mathrm{i}x_{j_0}$ of the function $\hat{\varepsilon}^{(0)}_{j_0}(x)$, which is the nearest pole to the real axis, and the expression (3.10) is simplified:

$$\hat{\varepsilon}^{+}_{j_0}(x)\simeq G_+(x)\left(\frac{-G_+(\mathrm{i}x_{j_0})}{x+\mathrm{i}x_{j_0}}\mathrm{i}\mathcal{M}_{j_0}\exp(-x_{j_0}B)+\frac{zH}{2}\,\frac{\mathrm{i}G_-(0)}{x+\mathrm{i}0}\right). \tag{3.12}$$

Here $\mathcal{M}_{j_0}=\mathrm{res}_{x=\mathrm{i}x_{j_0}}\,\hat{\varepsilon}^{(0)}_{j_0}(x)$ and x_{j_0} are given by (2.7).

It is not difficult to see that

$$\varepsilon_{j_0}(B)=\lim_{x\to\infty}\mathrm{i}x\hat{\varepsilon}^{+}_{j_0}(x). \tag{3.13}$$

From this equality and (3.3) one finds the relation between H and B

$$H=2\frac{G_+(\mathrm{i}x_{j_0})}{G_-(0)}\,\frac{\mathcal{M}_{j_0}}{z}\exp(-x_{j_0}B). \tag{3.14}$$

For $B\to\infty$ the ground-state energy (3.2) may be expressed in terms of the function $\hat{\varepsilon}^{+}_{j_0}(x)$

$$\begin{aligned}\mathscr{E}_0(H)-\mathscr{E}_0(0)&=-\frac{1}{p_0}\int_0^\infty \varepsilon^{(0)}_{j_0}(\lambda+B)\varepsilon_{j_0}(\lambda+B)\,\mathrm{d}\lambda\\ &\simeq-\frac{1}{p_0}\mathcal{M}_{j_0}\exp(-x_{j_0}B)\hat{\varepsilon}^{+}_{j_0}(\mathrm{i}x_{j_0}).\end{aligned} \tag{3.15}$$

Using representation (3.12) for the $\hat{\varepsilon}^{+}_{j_0}(x)$ we obtain the leading term of the asymptotics of $\mathscr{E}_0(H)$ at $H \to 0$:

$$\mathscr{E}_0(H) - \mathscr{E}_0(0) = -\frac{z^2}{8p_0 x_{j_0} \hat{B}^{(0,0)}_{j_0 j_0}(0)} H^2 + \dots \tag{3.16}$$

So to calculate the magnetic susceptibility $\chi = -\partial^2 \mathscr{E}_0(H)/\partial H^2$ at $H = 0$ one does not need the exact expression of $G_{\pm}(x)$. In (3.16) present only $\hat{B}^{(0,0)}_{j_0 j_0}(0)$ and for χ we have the following expression:

$$\text{(i)}\ \chi_{m_1} = \frac{p_0 - 1}{\pi} \qquad \text{(ii)}\ \chi_{\sigma-1} = \frac{1}{\pi}\frac{\sigma - 1}{p_0 - \sigma + 1} \qquad \text{(iii)}\ \chi_{m_2} = \frac{b_0}{\pi p_2}. \tag{3.17}$$

Iterating equation (3.4) one can obtain the next terms of the asymptotics of $\chi(H)$ at $H \to 0$ (see Yang and Yang 1966, Babujian 1982).

In the cases when the Dirac sea consists of more than one sort of string the task of calculating the magnetic susceptibility reduces to the matrix factorisation problem. In the present paper we do not consider this case.

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Appendix 1

It is convenient to define the kernels $B^{(\alpha,\beta)}_{jk}$ by their Fourier transforms. We consider only the case when $\varepsilon = (-1)^r$. The second case $\varepsilon = (-1)^{r+1}$ may be considered in a similar way in accordance with the formulae of appendix 3 from the paper by Kirillov and Reshetikhin (1987). The matrices $A^{(0,0)}_{jk}$, $A^{(1,0)}_{jk} = A^{(0,1)}_{kj}$ and $A^{(1,1)}_{jk}$ may be found in the abovementioned appendix. The matrices $\hat{B}^{(2,2)}_{jk}(x)$, $\hat{B}^{(2,1)}_{jk}(x)$ and $\hat{B}^{(2,0)}_{jk}(x)$ have the following form:

$$\hat{B}^{(2,2)}_{jl}(x) = \delta_{jl} - \frac{1}{2\cosh(p_{i-1}x)}(\delta_{j,l+1} + \delta_{j,l-1}) \tag{A1.1}$$

$$\hat{B}^{(2,1)}_{jk}(x) = \frac{\cosh[(q_k + q_{m_i})x]}{\cosh(p_i x)}\delta_{j,m_{i-1}-1} \qquad m_{i-1} \leq k < m_i \tag{A1.2}$$

$$\hat{B}^{(2,0)}_{jm_i}(x) = \frac{1}{2\cosh(p_{i-1}x)}\delta_{j,m_{i-1}-1} \tag{A1.3}$$

$$\hat{B}^{(2,0)}_{j,m_{i-2}}(x) = \frac{1}{2\cosh(p_{i-1}x)}\delta_{j,m_{i-2}+1} \tag{A1.4}$$

where $m_{i-2} < j, l < m_i$ and $i = r + 1 (\mathrm{mod}\ 2)$, $i \leq r + 1$. If $j, k > m_{r+1}$ the matrices $\hat{B}^{(2,2)}_{jk}(x)$, $\hat{B}^{(2,0)}_{jk}(x)$ have the following form:

$$\hat{B}^{(2,2)}_{m_i k}(x) = \delta_{m_i k} - \hat{S}_{i+1}(x)(-\delta_{m_i-1,k} + \delta_{m_i+1,k}) \tag{A1.5}$$

$$\hat{B}^{(2,2)}_{jk}(x) = \delta_{jk} - \hat{S}_{i+1}(x)(\delta_{j,k+1} + \delta_{j,k-1}) \qquad m_i < j < m_{i+1} - 1 \tag{A1.6}$$

$$\hat{B}^{(2,2)}_{m_{i+1}-1,k}(x)=(1-\hat{d}_{i+1}(x))\delta_{m_{i+1}-1,k}-\hat{S}_{i+1}(x)\delta_{m_{i+1}-2,k}-\hat{S}_{i+2}(x)\delta_{m_{i+1},k} \tag{A1.7}$$

$$\hat{B}^{(2,2)}_{m_{\alpha+1}-2,k}(x)=(1-\hat{S}^2_{\alpha+1}(x)\delta_{m_{\alpha+1}-2,k}-\hat{S}_{\alpha+1}(x)(\delta_{m_{\alpha+1}-3,k}+\delta_{m_{\alpha+1}-1,k}+\delta_{m_{\alpha+1},k}) \tag{A1.8}$$

$$\hat{B}^{(2,2)}_{m_{\alpha+1}-1,k}(x)=\delta_{m_{\alpha+1}-1,k}-\hat{S}_{\alpha+1}(x)\delta_{m_{\alpha+1}-2,k} \tag{A1.9}$$

$$\hat{B}^{(2,2)}_{m_{\alpha+1},k}(x)=\delta_{m_{\alpha+1},k}+\hat{S}_{\alpha+1}(x)\delta_{m_{\alpha+1}-2,k} \tag{A1.10}$$

$$\hat{B}^{(2,0)}_{j,m_{i+1}}(x)=\hat{S}_{r+1}(x)\delta_{j,m_{r+1}}\delta_{i,r} \tag{A1.11}$$

$$\hat{S}_i(x)=\frac{2}{2\cosh(p_ix)}\qquad \hat{d}_i(x)=\frac{\cosh[(p_i-p_{i+1})x]}{2\cosh(p_ix)\cosh(p_{i+1}x)}. \tag{A1.12}$$

These formulae are written for the rational parameter $p_0=\pi/\gamma=[b_0,b_1,b_2,\ldots,b_{\alpha+1}]$, $b_i\geqslant 2$.

Appendix 2

Let us consider the function

$$L(r,\theta)=\operatorname{Re} L(r\,e^{i\theta})$$
$$=-\frac{1}{2}\int_0^r\frac{\log(1-2\rho\cos\theta+\rho^2)}{\rho}\,d\rho+\tfrac{1}{4}\log|r|\log(1-2r\cos\theta+r^2)$$

where $L(x)$ is the Rogers dilogarithmic function defined by (2.17).

The function $L(r,\theta)$ has the following properties:

$$L(-1,\theta)=\tfrac{1}{4}\theta^2-\tfrac{1}{12}\pi^2\qquad 0\leqslant\theta\leqslant\pi \tag{A2.1}$$

$$L(1,\theta)=\tfrac{1}{4}(\pi-\theta)^2-\tfrac{1}{12}\pi^2\qquad 0\leqslant\theta\leqslant\pi \tag{A2.2}$$

$$L\left(-\frac{\sin\varphi}{\sin\theta},\varphi+\theta\right)+L\left(-\frac{\sin\theta}{\sin\varphi},\varphi+\theta\right)=\tfrac{1}{2}(\varphi+\theta)^2-\tfrac{1}{6}\pi^2 \tag{A2.3}$$

$$L\left(\frac{\sin^2\varphi}{\sin^2\theta}\right)=2\theta\varphi+2L\left(-\frac{\sin(\varphi-\theta)}{\sin\theta},\varphi\right)-2L\left(-\frac{\sin\varphi}{\sin\theta},\varphi+\theta\right). \tag{A2.4}$$

The formulae (A2.1)–(A2.4) may be proved by the differentiation of the left- and right-hand sides of these equalities. For example, let us prove (A2.4). It is easy to see that

$$dL(r,\theta)=\left(-\frac{1}{4}\frac{\log(1-2r\cos\theta+r^2)}{r}+\frac{1}{2}\log\frac{r-\cos\theta}{1-2r\cos\theta+r^2}|r|\right)dr$$
$$+\left[-\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)+\frac{1}{2}\log\frac{r\sin\theta}{1-2r\cos\theta+r^2}|r|\right]d\theta.$$

So, we have

$$\frac{d}{d\varphi}L\left(-\frac{\sin\varphi}{\sin\theta},\varphi+\theta\right)=-\frac{1}{2}\cot\varphi\log\left(\frac{\sin(\varphi+\theta)}{\sin\theta}\right)+\cot(\varphi+\theta)\log\left(\frac{\sin\varphi}{\sin\theta}\right)+\varphi.$$

Similarly, one can prove that

$$\frac{d}{d\varphi}L\left(\frac{\sin^2\theta}{\sin^2\varphi}\right)=\cot\varphi\log\left(\frac{\sin(\varphi-\theta)\sin(\varphi+\theta)}{\sin^2\theta}\right)$$
$$-\frac{\sin 2\varphi}{\sin(\varphi-\theta)\sin(\varphi+\theta)}\log\left(\frac{\sin\varphi}{\sin\theta}\right).$$

Comparing these formulae one can see that the derivatives of the left- and right-hand sides of (A2.4) over φ coincide. The equality (A2.4) may now be deduced from (A2.1). From (A2.4) it follows that

$$\sum_{k=1}^{m} L\left(\frac{\sin^2\theta}{\sin^2(\varphi+k\theta)}\right)=2\theta\left(m\varphi+\frac{m(m+1)}{2}\theta\right)+2L\left(-\frac{\sin\varphi}{\sin\theta},\varphi+\theta\right)$$
$$-2L\left(-\frac{\sin(\varphi+m\theta)}{\sin\theta},\varphi+(m+1)\theta\right)\qquad 0\leqslant\varphi+(m+1)\theta\leqslant\pi. \quad \text{(A2.5)}$$

Let us take $\varphi=0$, $(m+1)\theta=\pi$, $n=m+1$ in this equality. After some easy transformations we obtain the equality (2.34).

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